Probability

- Cardano
- Mathematical model (random experiment)
- Sample space ($\mathcal{S}$), random variable (X), events
- Probability distribution: $P : \mathcal{F} \rightarrow [0,1]$ where:

$$P(\mathcal{S}) = 1$$

  and if A and B are mutually exclusive events (i.e., $A \cap B = \emptyset$), then

$$P(A \cup B) = P(A) + P(B)$$

- Odds vs. probability

Theorems

- The probability of an event and its complement must add up to one:

$$P(\overline{A}) = 1 - P(A)$$

- The probability of an impossible event is zero:

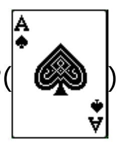
$$P(\emptyset) = 0$$

- In the case where all outcomes are equally likely, the probability of any one outcome is:

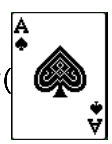
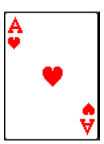
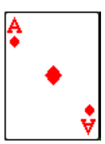
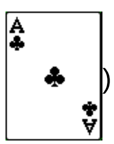
$$P(s \in \mathcal{S}) = \frac{1}{|\mathcal{S}|}$$

Probability

- Take a standard 52 card deck. Then the probability of drawing any particular card is:

$P($  $) = \frac{1}{52}$

- What's the probability of drawing an ace?

$P($  or  or  or  $) = \frac{1+1+1+1}{52} = \frac{1}{13}$

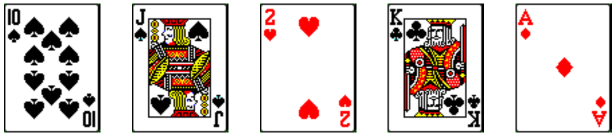
- Sum Rule: $P(A \text{ or } B) = P(A) + P(B)$ when A and B are mutually exclusive

Draw Poker

- Each player antes and gets five cards
- Players can either *fold* and drop out of the game, or *bet*/*call*/*raise*, adding more money to the pot
- Players still in the game can trade some of their cards for new ones
- Again, players can either fold or call
- The player who hasn't folded and is holding the best hand wins the entire pot (antes + first round bets + second round bets)
- Hands are ranked: straight flush, four of a kind, full house (pair and three), flush, straight, three of a kind, two pair, pair, high card

Draw Poker

- Drawing to an inside straight:



- Four cards (any queen) would complete the straight, and there 47 cards left in the deck. So,

$$P(\text{straight}) = \frac{4}{47} \approx 0.09$$

- The odds are 43 : 4 or about 10.75 : 1 against making the straight.

Probability

- Three card monte:

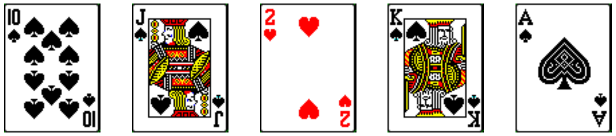
- Guess which card is the red queen:

$$P(\text{queen}) = \frac{1}{3}$$

- If the house is paying $2 for your $1, then it's a fair game, right?

Draw Poker

- Recall that the probability of hitting an inside straight is $\frac{4}{47}$, or 10.75 : 1 odds.
- Drawing to an inside straight flush:



- Four cards (any queen) would complete the straight
- Nine cards (any spade) would complete the flush
- One card (queen of spades) would complete the straight flush

Draw Poker

- The chance of hitting a straight or a flush is:

$$\begin{aligned} P(\text{straight} \cup \text{flush}) &= P(\text{straight}) + P(\text{flush}) - P(\text{straight flush}) \\ &= \frac{4}{47} + \frac{9}{47} - \frac{1}{47} \\ &= \frac{12}{47} \end{aligned}$$

- The chance of hitting a straight or a flush or a straight flush is also $\frac{12}{47} \approx 0.26$.
- The odds are 35 : 12 or about 2.92 : 1 against making a hand.

Counting

- What's the probability of drawing five particular cards?

- First, how many ways are there to draw five cards without replacement?

$$52 \times 51 \times 50 \times 49 \times 48 = 311,875,200$$

Counting

- In general, the number of ways to select k objects in order from a set of n is without replacement:

$${}_nP_k = \frac{n!}{(n-k)!}$$

- For drawing five cards from a deck of 52, we have:

$$\begin{aligned} {}_{52}P_5 &= \frac{52!}{(52-5)!} \\ &= \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47 \times \cdots \times 1}{47 \times \cdots \times 1} \\ &= 52 \times 51 \times 50 \times 49 \times 48 \\ &= 311,875,200 \end{aligned}$$

Counting

- Back to this hand:

- There are 311,875,200 ways of being dealt five cards, and this is one of them.

- So, the probability of getting these cards *in this order* is:

$$P(\text{cards}) = \frac{1}{311,875,200}$$

Counting

- But: order doesn't matter! We would be just as happy to get this hand:

- How many different ways can a five card hand can be dealt out?

$$\begin{aligned} {}_5P_5 &= \frac{5!}{(5-5)!} \\ &= \frac{5 \times 4 \times 3 \times 2 \times 1}{1} \\ &= 120 \end{aligned}$$

Counting

- If each unordered hand can be dealt out in 120 different orders, then the total number of distinct poker hands is:

$$311{,}875{,}200 \div 120 = 2{,}598{,}960$$

- The probability of getting dealt any one particular hand is:

$$P(\text{hand}) = \frac{1}{2{,}598{,}960}$$

Since there are four suits flushes, the chances of getting dealt a pat royal flush is:

$$P(\text{royal}) = \frac{4}{2{,}598{,}960} = \frac{1}{649{,}740}$$

Counting

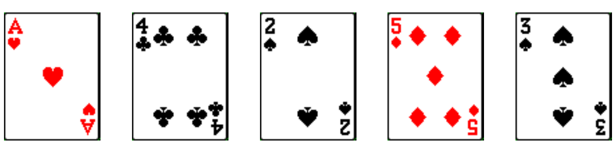
- In general, the number of unordered ways to select k objects from a set of n is without replacement is:

$$\begin{aligned} \binom{n}{k} &= \frac{{}_nP_k}{{}_kP_k} \\ &= \frac{\frac{n!}{(n-k)!}}{\frac{k!}{0!}} \\ &= \frac{n!}{k!(n-k)!} \end{aligned}$$

- Known as the *binomial coefficient*
- This also sometimes written ${}_nC_k$ and pronounced “n choose k”

Counting

- What’s the probability of drawing a straight?



- Any card except J, Q, or K can start a straight. Then, for each position there are four cards that continue the straight:

$$40 \times 4 \times 4 \times 4 \times 4 = 10{,}240$$

Counting

- But, some of these hands are straight flushes:

$$40 \times 1 \times 1 \times 1 \times 1 = 40$$

- So, the probability of getting dealt a pat straight is:

$$\begin{aligned} P(\text{straight}) &= \frac{10{,}240 - 40}{2{,}598{,}960} \\ &= \frac{10{,}200}{2{,}598{,}960} \\ &= 0.00392 \end{aligned}$$

- The odds are $2{,}588{,}760 : 10{,}200$ or about $254 : 1$ against.

Summary

- Sum rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Sampling with and without replacement
- Ordered samples:

$${}_nP_k = \frac{n!}{(n-k)!}$$

- Unordered samples:

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$$

Conditional probability

- Often the sample space $\mathcal{S}$ is too large to be interesting, or even known. Instead, we only care about a subset of possible outcomes that have some relevant property.
- The *conditional probability* of an event B given event A is:

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{|A \cap B|}{|\mathcal{S}|} \times \frac{|\mathcal{S}|}{|A|} \\ &= \frac{|A \cap B|}{|A|} \end{aligned}$$

- The *joint probability* $P(AB)$ is the probability of events A and B occurring together: $P(A \cap B)$

Conditional probability

- Someone you have just met has two children. What is the probability that they are both girls?
- Someone you have just met has two children. At least one of them is a girl. What is the probability that they are both girls?
- Someone you have just met has two children. The oldest is a girl. What is the probability that they are both girls?