

# Probability

- Cardano
- Mathematical model (random experiment)
- Sample space ( $\mathcal{S}$ ), random variable ( $X$ ), events
- Probability distribution:  $P : \mathcal{F} \rightarrow [0, 1]$  where:

$$P(\mathcal{S}) = 1$$

and if  $A$  and  $B$  are mutually exclusive events (i.e.,  $A \cap B = \emptyset$ ), then

$$P(A \cup B) = P(A) + P(B)$$

- Odds vs. probability

# Theorems

- The probability of an event and its complement must add up to one:

$$P(\bar{A}) = 1 - P(A)$$

- The probability of an impossible event is zero:

$$P(\emptyset) = 0$$

- In the case where all outcomes are equally likely, the probability of any one outcome is:

$$P(s \in \mathcal{S}) = \frac{1}{|\mathcal{S}|}$$

# Probability

- Take a standard 52 card deck. Then the probability of drawing any particular card is:

$$P(\text{Ace of Spades}) = \frac{1}{52}$$

- What's the probability of drawing an ace?

$$P(\text{Ace of Spades or Ace of Hearts or Ace of Diamonds or Ace of Clubs}) = \frac{1+1+1+1}{52} = \frac{1}{13}$$

- Sum Rule:  $P(A \text{ or } B) = P(A) + P(B)$  when  $A$  and  $B$  are mutually exclusive

# Draw Poker

- Each player antes and gets five cards
- Players can either *fold* and drop out of the game, or *bet/call/raise*, adding more money to the pot
- Players still in the game can trade some of their cards for new ones
- Again, players can either fold or call
- The player who hasn't folded and is holding the best hand wins the entire pot (antes + first round bets + second round bets)
- Hands are ranked: straight flush, four of a kind, full house (pair and three), flush, straight, three of a kind, two pair, pair, high card

# Draw Poker

- Drawing to an inside straight:



- Four cards (any queen) would complete the straight, and there are 47 cards left in the deck. So,

$$P(\text{straight}) = \frac{4}{47} \approx 0.09$$

- The odds are 43 : 4 or about 10.75 : 1 against making the straight.

# Probability

- Three card monte:



- Guess which card is the red queen:

$$P(\text{queen}) = \frac{1}{3}$$

- If the house is paying \$2 for your \$1, then it's a fair game, right?

# Draw Poker

- Recall that the probability of hitting an inside straight is  $\frac{4}{47}$ , or 10.75 : 1 odds.
- Drawing to an inside straight flush:



- Four cards (any queen) would complete the straight
- Nine cards (any spade) would complete the flush
- One card (queen of spades) would complete the straight flush

# Draw Poker

- The chance of hitting a straight or a flush is:

$$\begin{aligned}P(\text{straight} \cup \text{flush}) &= P(\text{straight}) + P(\text{flush}) - P(\text{straight flush}) \\ &= \frac{4}{47} + \frac{9}{47} - \frac{1}{47} \\ &= \frac{12}{47}\end{aligned}$$

- The chance of hitting a straight or a flush or a straight flush is also  $\frac{12}{47} \approx 0.26$ .
- The odds are 35 : 12 or about 2.92 : 1 against making a hand.

# Counting

- What's the probability of drawing five particular cards?



- First, how many ways are there to draw five cards without replacement?

$$52 \times 51 \times 50 \times 49 \times 48 = 311,875,200$$

# Counting

- In general, the number of ways to select  $k$  objects in order from a set of  $n$  is without replacement:

$${}_n P_k = \frac{n!}{(n-k)!}$$

- For drawing five cards from a deck of 52, we have:

$$\begin{aligned} {}_{52} P_5 &= \frac{52!}{(52-5)!} \\ &= \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47 \times \cdots \times 1}{47 \times \cdots \times 1} \\ &= 52 \times 51 \times 50 \times 49 \times 48 \\ &= 311,875,200 \end{aligned}$$

# Counting

- Back to this hand:



- There are 311,875,200 ways of being dealt five cards, and this is one of them.
- So, the probability of getting these cards *in this order* is:

$$P(\text{cards}) = \frac{1}{311,875,200}$$

# Counting

- But: order doesn't matter! We would be just as happy to get this hand:



- How many different ways can a five card hand can be dealt out?

$$\begin{aligned} {}_5P_5 &= \frac{5!}{(5-5)!} \\ &= \frac{5 \times 4 \times 3 \times 2 \times 1}{1} \\ &= 120 \end{aligned}$$

# Counting

- If each unordered hand can be dealt out in 120 different orders, then the total number of distinct poker hands is:

$$311,875,200 \div 120 = 2,598,960$$

- The probability of getting dealt any one particular hand is:

$$P(\text{hand}) = \frac{1}{2,598,960}$$

Since there are four suits flushes, the chances of getting dealt a pat royal flush is:

$$P(\text{royal}) = \frac{4}{2,598,960} = \frac{1}{649,740}$$

# Counting

- In general, the number of unordered ways to select  $k$  objects from a set of  $n$  is without replacement is:

$$\begin{aligned}\binom{n}{k} &= \frac{{}_n P_k}{{}_k P_k} \\ &= \frac{\frac{n!}{(n-k)!}}{\frac{k!}{0!}} \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

- Known as the *binomial coefficient*
- This also sometimes written  ${}_n C_k$  and pronounced “ $n$  choose  $k$ ”

# Counting

- What's the probability of drawing a straight?



- Any card except J, Q, or K can start a straight. Then, for each position there are four cards that continue the straight:

$$40 \times 4 \times 4 \times 4 \times 4 = 10,240$$

# Counting

- But, some of these hands are straight flushes:

$$40 \times 1 \times 1 \times 1 \times 1 = 40$$

- So, the probability of getting dealt a pat straight is:

$$\begin{aligned} P(\text{straight}) &= \frac{10,240 - 40}{2,598,960} \\ &= \frac{10,200}{2,598,960} \\ &= 0.00392 \end{aligned}$$

- The odds are 2,588,760 : 10,200 or about 254 : 1 against.

# Summary

- Sum rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Sampling with and without replacement

- Ordered samples:

$${}_n P_k = \frac{n!}{(n-k)!}$$

- Unordered samples:

$$\binom{n}{k} = {}_n C_k = \frac{n!}{k!(n-k)!}$$

# Conditional probability

- Often the sample space  $\mathcal{S}$  is too large to be interesting, or even known. Instead, we only care about a subset of possible outcomes that have some relevant property.
- The *conditional probability* of an event  $B$  given event  $A$  is:

$$\begin{aligned}P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{|A \cap B|}{|\mathcal{S}|} \times \frac{|\mathcal{S}|}{|A|} \\ &= \frac{|A \cap B|}{|A|}\end{aligned}$$

- The *joint probability*  $P(AB)$  is the probability of events  $A$  and  $B$  occurring together:  $P(A \cap B)$

# Conditional probability

- Someone you have just met has two children. What is the probability that they are both girls?
- Someone you have just met has two children. At least one of them is a girl. What is the probability that they are both girls?
- Someone you have just met has two children. The oldest is a girl. What is the probability that they are both girls?