Probabilistic models

- Decision trees, instance-based learning, and transformation-based learning are called *non-parametric* methods because they don't use an explicit probabilistic model
- *Parametric* machine learning methods assume a particular (typically probabilistic) model
- Parametric methods (usually) search a much more restrictive hypothesis space than non-parametric methods $\rightarrow$ large bias, small variance

Probabilistic models

- Suppose we have a representation of an instance as feature vector x and we want to predict its class c
- If we have a way of modeling $P(c|x)$, *Bayes Decision Rule* says our predicted $\hat{c}$ should be:

$$\hat{c} = \operatorname*{argmax}_{c \in C} P(c|x)$$

- This minimizes the expected error:

$$\begin{aligned} P(\text{error}|x) &= 1 - P(\hat{c}|x) \\ P(\text{error}) &= \sum_x P(\text{error}|x)\, P(x) \end{aligned}$$

Probabilistic models

- There are two ways of applying the Bayes decision rule
- A *discriminative* (aka *diagnostic*) method directly models $P(c|x)$
- More commonly, a *generative* (aka *sampling*) method is used, which models the joint distribution $P(x,c)$ and uses Bayes rule:

$$\begin{aligned}
\hat{c} &= \operatorname*{argmax}_{c \in C} P(c|x) \\
&= \operatorname*{argmax}_{c \in C} \frac{P(x|c)\,P(c)}{P(x)} \\
&= \operatorname*{argmax}_{c \in C} P(x|c)\,P(c) \\
&= \operatorname*{argmax}_{c \in C} P(x,c)
\end{aligned}$$

Probabilistic models

- If Bayes decision rule minimizes error, why do we still make mistakes?
- Overlapping classification functions (where $P(c|x) \neq 1$) can never be learned perfectly
- The classifier only works as well as our model – if our model $P(c|x)$ is inaccurate, then we'll make the wrong decisions
- We need some way of constructing, evaluation, and selecting probability models

Baseline classifier

- We often compute a ‘baseline’ for a classification task by simply assigning the most frequent class to each instance:

$$\hat{c} = \underset{c \in C}{\operatorname{argmax}} P(c)$$

- Here we assume that $P(c|x) = P(c)$, i.e., X and C are independent
- The extra error a baseline classifier makes is:

$$\sum_{x} P(x) \left[P(x,c) - P(x) P(c) \right]$$

Bayes Optimal Classifiers

- Call a particular model h, chosen from the hypothesis space H.
- The *maximum likelihood* hypothesis selects:

$$\hat{c} = \underset{c \in C, h \in H}{\operatorname{argmax}} P(c|x,h)\, P(d|h)$$

- The *maximum a posteriori* hypothesis selects:

$$\hat{c} = \underset{c \in C, h \in H}{\operatorname{argmax}} P(c|x,h)\, P(d|h)\, P(h)$$

- Both of these commit us to choosing one h, which may or may not wind up being the best choice

Bayes Optimal Classifiers

- The *Bayes Optimal Classifier* selects:

$$\hat{c} = \underset{c \in C}{\operatorname{argmax}} \sum_{h \in H} P(c|x,h)\, P(d|h)\, P(h)$$

- We remove the dependence on a particular h by averaging over all possible hs
- This is almost always impossible to apply in practice, but it can used to establish a lower bound on the error rate
- We can also sometimes approximate it, e.g., by randomly drawing h from the posterior distribution $P(d|h)\, P(h)$

Rev. Thomas Bayes (1702–1761)

Rev. Thomas Bayes (1702–1761)

Naive Bayes classifiers

- To apply a generative Bayesian classifier, we need $P(x, c)$
- We can break this down into two parts: the *class prior* $P(c)$, and a likelihood $P(x|c)$
- The class priors are easy to estimate from training data:

$$\hat{P}(c) = \frac{\text{\# of instances in class } c}{\text{\# of instances}}$$

- This won't work for $P(x|c)$, since any particular feature vector x is unlikely to turn up in the training data:

$$\hat{P}(x|c) = \frac{\text{\# of instances of } x \text{ in } c}{\text{\# of instances in } c} \approx \frac{0}{\text{\# of instances of } x \text{ in } c}$$

Naive Bayes classifiers

- To get a better estimate of $P(x|c)$, we can make the simplifying assumption that each of the dimensions x_i in x are independent, so that:

$$P(x|c) = \prod_i P(x_i|c)$$

- Now we only need to get estimates of $P(x_i|c)$ from the data for each x_i, which we can do in the usual way:

$$\hat{P}(x_i|c) = \frac{\#\text{ of instances of } x_i \text{ in } c}{\#\text{ of instances in } c}$$

- Both $\hat{P}(c)$ and $\hat{P}(x_i|c)$ can be estimated using whatever tricks we have available

Naive Bayes classifiers

- The naive Bayes classifier selects the class $\hat{c}$ such that:

$$\hat{c} = \underset{c \in C}{\operatorname{argmax}} P(c) \prod_i P(x_i|c)$$

- Naive Bayes classifiers have been used primarily for classifying texts (Maron 1961)
- We treat a text as a *set* or *bag of words*, an unordered collection of all the words that appear in the text
- "We treat a text as a set or bag of words" $\equiv \{$ a, a, as, bag, of, or, set, text, treat, we, words $\}$

Naive Bayes classifiers

- Ignoring word order in the feature representation removes the most obvious syntactic dependencies between words

$$P(\text{the})\,P(\text{book}) \neq P(\text{the book})$$

- There are still semantic dependencies:

$$P(\text{tackle})\,P(\text{touchdown}) \neq P(\text{tackle}, \text{touchdown})$$

- And, multiple occurrences of words are probably not independent

Text classification

- Text classification can be useful for information retrieval and natural language processing tasks
  - indexing
  - message routing
  - summarization
- Text classification also plays a role in linguistic research
  - authorship identification
  - genre studies
  - forensic linguistics
  - sociolinguistics
- A combination of the two makes the WWW available as a resource for research

Feature selection

- A straight bag-of-words model leads to positing a very large number of features
- Some of those features will not be relevant for the task (stop words)
- Many of the features will appear relevant, but won't be: we can't avoid the Curse of Dimensionality
- So, we want to select a subset of features which appear promising, usually by information gain

Multivariate Bernoulli event model

- If we represent a document as a *set* of words, then each feature x_i is a Bernoulli variable, where:

$$P(x_i|c_j) = P(x_i = 1|c_j)^{x_i} \left(1 - P(x_i = 1|c_j)\right)^{1-x_i}$$

- If there are v words in the vocabulary, a document is constructed by flipping v coins

- Call $p_{ij} = P(x_i = 1|c_j)$. Substituting this in, we get:

$$\begin{aligned} P(c_j|x) &= \frac{P(c_j)\prod_i P(x_i|c_j)}{P(x)} \\ &= \frac{P(c_j)\prod_i p_{ij}^{x_i}(1-p_{ij})^{1-x_i}}{P(x)} \end{aligned}$$

Multivariate Bernoulli event model

- And taking the log gives us:

$$\begin{aligned}
\log P(c_j|x) &= \log P(c_j) + \sum_i x_i \log p_{ij} + \sum_i (1-x_i)\log(1-p_{ij}) - \log P(x) \\
&= \log P(c_j) + \sum_i x_i \log p_{ij} + \sum_i \log(1-p_{ij}) - \sum x_i \log(1-p_{ij}) - \\
&\qquad \log P(x) \\
&= \log P(c_j) + \sum_i x_i \log \frac{p_{ij}}{1-p_{ij}} + \sum_i \log(1-p_{ij}) - \log P(x)
\end{aligned}$$

- Suppose we only have two classes. Then $P(c_1|x) = 1 - P(c_2|x)$, and the posterior log odds are:

$$\log \frac{P(c_1|x)}{1-P(c_1|x)} = \sum_i x_i \log \frac{p_{i1}(1-p_{i2})}{(1-p_{i1})p_{i2}} + \sum_i \log \frac{1-p_{i1}}{1-p_{i2}} + \log \frac{P(c_1)}{1-P(c_1)}$$

Multivariate Bernoulli event model

- Under this *binary independence model*, the parameters p_{ij} can be estimated via:

$$\hat{p}_{ij} = \frac{\#\text{ of documents containing } x_i \text{ in } c_j}{\#\text{ of documents in class } c_j}$$

- Note that this doesn't take into account the length of the document
- It also doesn't take into account the number of times a word appears in a document

Multinomial event model

- If instead we represent a document as a *bag* of words, then we can model a document as a sequence of random draws from a multinomial distribution
- The probability of picking word w_i if the document class is c_j once is $P(w_i|c_j)$
- The probability of picking word w_i x_i times in a row is $P(w_i|c_j)^{x_i}$
- The probability of drawing a collection of words *in that order* is:

$$\prod_i P(w_i|c_i)^{x_i}$$

Multinomial event model

- This underestimates $P(x|c_j)$, since lots of ordered sequences correspond to the same bag of words
- How many different ways are there to draw word w_1 x_1 times, word w_2 x_2, and so on?
- We can use the *multinomial coefficient*:

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots} &= \binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \cdots \\ &= \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times \cdots \\ &= \frac{n!}{n_1! n_2! \cdots}\end{aligned}$$

Multinomial event model

- So, if we draw $N = \sum_i x_i$ words, we have:

$$\begin{aligned} P(x|c_j) &= \binom{N}{x_1, x_2, \ldots} \prod P(w_i|c_j)^{x_i} \\ &= N! \prod \frac{P(w_i|c_j)^{x_i}}{x_i!} \end{aligned}$$

- To be completely correct, we also need to think about the probability of finding a document of a particular length:

$$P(x|c_j) = P(N|c_j)(\sum_i x_i)! \prod \frac{P(w_i|c_j)^{x_i}}{x_i!}$$

but in practice this can be hard to do.

Multinomial event model

- The parameters of the multinomial model are the individual word probabilities $P(w_i|c_j)$
- Since these are the parameters of a multinomial distribution, we need to maintain:

$$\sum_i P(w_i|c_j) = 1$$

- We can estimate those from training data as:

$$\hat{P}(w_i|c_j) = \frac{\text{\# of times } w_i \text{ occurs in documents in } c_j}{\text{\# of words in documents in class } c_j}$$

- As always, smoothing is important

Text classification

- The multinomial model takes word frequencies and document length into account, but treats multiple occurrences of a word as independent events
- McCallum and Nigam (1998) compare the two event models
- Multinominal model almost always outperforms multivariate Bernoulli model, by 25% or so
- The multinominal model handles large vocabulary sizes much better
- It’s easier to see how to add non-text features and to account for limited inter-dependencies using a multivariate Bernoulli model

Naive Bayes classifiers

- Despite its obvious limitations, naive Bayes text classifiers work quite well
- Lewis and Ringuette 1994 ‘breakeven point’ for naive Bayes very close to decision trees
- In other work, naive Bayes scores close to, but consistently worse than, more sophisticated methods
- Since naive Bayes is pretty good, and it’s easy to implement, it is very widely used

Naive Bayes classifiers

- Paul Graham wrote an article on naive Bayes classifiers for filtering junk mail, which has become a standard method

```
Free CableTV!No more pay!%RND_SYB

requisite silt administer orphanage teach
hypothalamus diatomic conflict atlas moser
cofactor electret coffin diversionary solicitous
becalm absent satiable blurb mackerel sibilant
tehran delivery germicidal barometer falmouth
capricorn
```

Naive Bayes classifiers

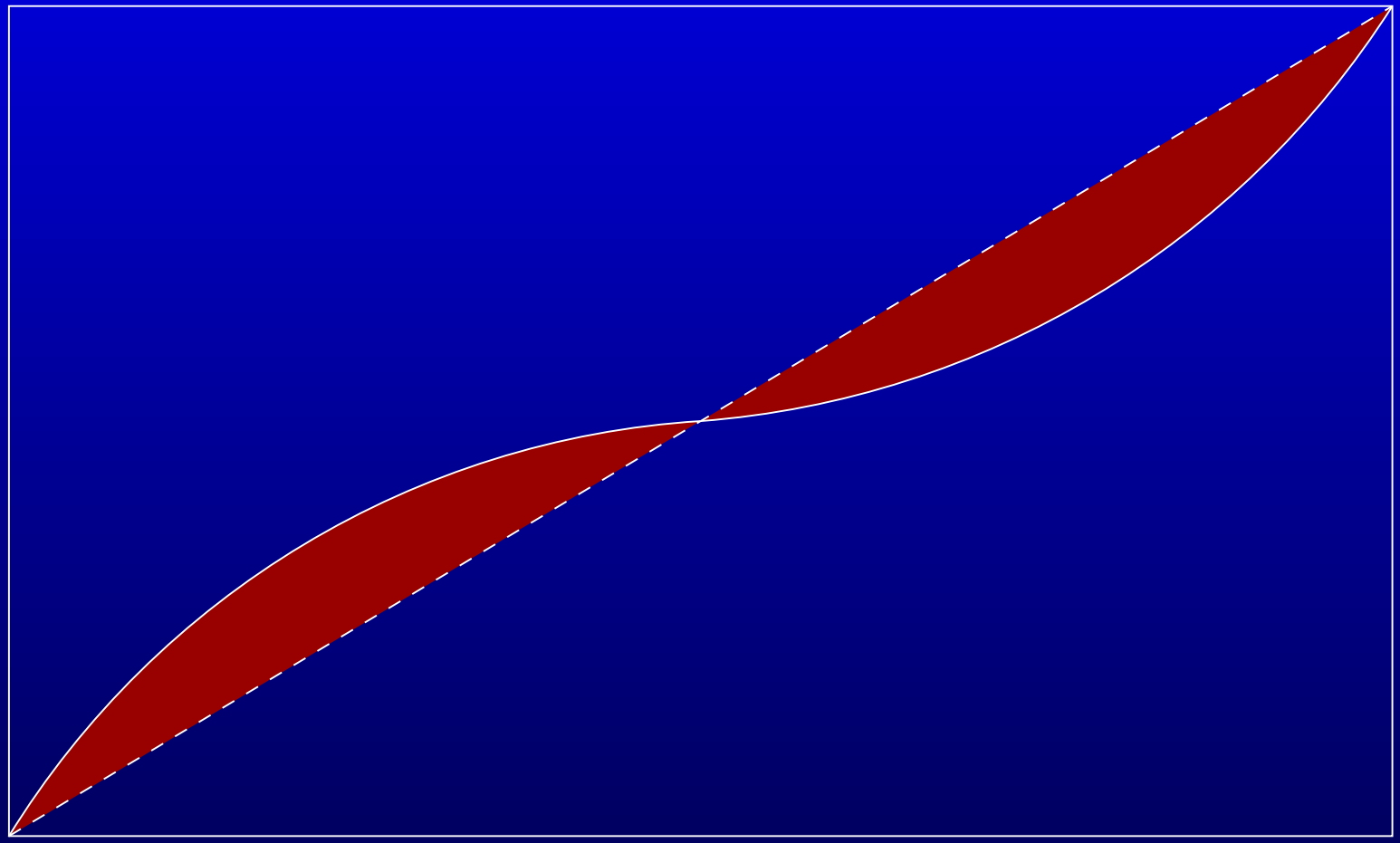
- Maron (1961):

  It is feasible to have a computing machine read a document and to decide automatically the subject category to which the item in question belongs. No real intellectual breakthroughs are required before a machine will be able to index rather well. Just as in the case of machine translation of natural language, the road is gradual but, by and large, straightfoward.

Zero-one loss

- Given its obvious deficiencies, why does naive Bayes work as well as it does?
- Its probability estimates are only as good as the independence assumptions are valid (i.e., not very)
- But, we don’t evaluate a naive Bayes classifier on its probability estimates
- Instead, we measure its misclassification error, or *zero-one loss*
- The two measures need not be closely related

Zero-one loss



Zero-one loss

- If the features are independent, then naive Bayes is optimal under zero-one loss
- Domingos and Pazzani (1997) evaluate naive Bayes on problems from the UCI repository, and find it often performs very well, but sometimes it performs badly
- They then used mutual information to measure the pairwise dependencies between features
- There was no clear relationship between the validity of independence assumptions and the performance of naive Bayes

Zero-one loss

- Suppose there are two classes, and let $p = P(c_1|x)$, $r = P(c_1)\prod_i P(x_i|c_1)$ and $s = r = P(c_2)\prod_i P(x_i|c_2)$
- For any instance x, naive Bayes is optimal under zero-one loss if and only $(p \geq \frac{1}{2} \wedge r \geq s) \vee (p \leq \frac{1}{2} \wedge r \leq s)$
- That means that naive Bayes is optimal under zero-one loss for half the volume of the space of possible values of (p, r, s)!
- The naive Bayes probabilities are optimal only along the line where the planes $r = p$ and $s = 1 - p$ intersect

Zero-one loss

- A necessary condition: naive Bayes can only be optimal (for discrete features) for concepts that are linearly separable
- For discrete features, combinations of variables by $\wedge$, $\vee$, and $\neg$ are linearly separable
- This isn't a sufficient condition, since there are linearly separable concepts which naive Bayes performs poorly on (m-of-n concepts)
- Naive Bayes is optimal for conjunctions of features and for disjunctions of features
- This points to one way to improve naive Bayes: introduce new features which are disjunctions (or conjunctions) of other features

Zero-one loss

- Even when naive Bayes is not optimal, it may outperform other methods with greater representational power (e.g., C4.5)
- Zero-one loss is relatively insensitive to *bias,* but can be highly sensitive to *variance*
- When there isn't enough training data, a high bias, low variance learner will give a lower zero-one loss than a low bias, high variance learner
- We've seen this before: a simple model can outperform a more complex one, even when the assumptions of the simple model are false