Probabilistic classifiers

- Bayes Decision Rule minimizes expected error:

$$\hat{c} = \operatorname*{argmax}_{c \in C} P(c|x)$$

- We use a *generative* model $P(x, c)$ plus Bayes' Theorem:

$$\begin{aligned}\hat{c} &= \operatorname*{argmax}_{c \in C} P(c|x) \\ &= \operatorname*{argmax}_{c \in C} \frac{P(x|c)\,P(c)}{P(x)} \\ &= \operatorname*{argmax}_{c \in C} P(x|c)\,P(c) \\ &= \operatorname*{argmax}_{c \in C} P(x, c)\end{aligned}$$

Naive Bayes classifiers

- We can split $P(x, c)$ into two parts: the class prior $P(c)$, and a likelihood $P(x|c)$
- It's easy to get reasonable estimates of $P(c)$ from training data, but not $P(x|c)$
- Instead, we assume that the individual features in x are independent, so:

$$P(x|c) = \prod_i P(x_i|c)$$

- Now the decision rule becomes:

$$\hat{c} = \operatorname*{argmax}_{c \in C} P(c) \prod_i P(x_i|c)$$

Naive Bayes classifiers

- Naive Bayes classifiers work well even when features aren't independent
- But, the "naive Bayes" assumption is clearly wrong – can we do without it?
- If we know all the $P(x_i|c)$'s but not their dependencies, is it possible to construct $P(x|c)$?
- Yes, in fact, there are lots of ways to do it: the problem is *ill-posed*

Maximum Entropy

- This is a general problem: how do we pick a probability distribution given possibly incomplete information?
- Our probability estimates should reflect what we know *and what we don't know*: ignorance is preferable to error
- Shannon's entropy is a measure of ignorance
- Jaynes (1957): "The least informative probability distribution maximizes the entropy S subject to known constraints."

Principle of Insufficient Reason

- Remember Bernoulli's *Principle of Insufficient Reason*: if we have n outcomes and don't know anything else, then say each outcome has a probability of $\frac{1}{n}$
- Suppose we have a coin (with two sides). All we know is:

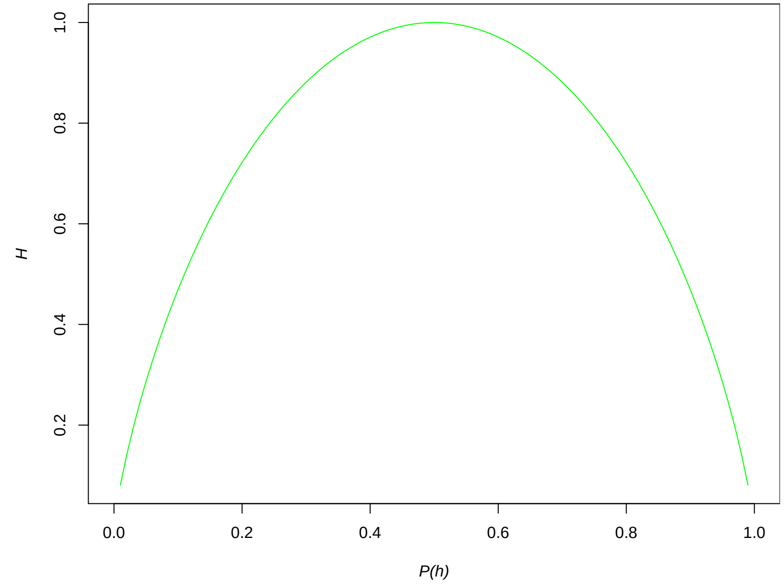
$$P(h) + P(t) = 1$$

- The entropy H is:

$$\begin{aligned} H &= -(P(h)\log P(h) + P(t)\log P(t)) \\ &= -(P(h)\log P(h) + (1 - P(h))\log(1 - P(h))) \end{aligned}$$

- If $P(h) = 0.5$, then $H = \log_2 1 = 1$ bit

Principle of Insufficient Reason



Wallis derivation

- Why maximize entropy? Shannon and Jaynes show that other measures run into inconsistencies.
- Another argument (what Jaynes calls the "Wallis derivation") based on a procedure for 'fairly' constructing a distribution given some constraints
- Divide the available probability mass into n quanta, each of magnitude $\delta = \frac{1}{n}$, and randomly assign them to the m possible outcomes.
- If outcome i gets n_i quanta, then we say its probability is $p_i = n_i\,\delta = \frac{n_i}{n}$
- If the resulting distribution fits the known constraints, we're done. Otherwise, we reject it and try again.

Wallis derivation

- For this to give good results, n has to be much larger than m, and we might need a lot of attempts before we get a distribution that fits the constraints
- So, instead, let's find the distribution which is most likely to come up
- The probability of any particular assignment is given by the multinomial distribution:

$$P(n_1, \ldots, n_m) = \binom{n}{n_1, \ldots, n_m} m^{-n} = \frac{n!}{n_1! \cdots n_m!} m^{-n}$$

- So, the assignment which we are most likely to come up with using this fair procedure is the one that maximizes:

$$W = \frac{n!}{n_1! \cdots n_m!}$$

Wallis derivation

- Instead of maximizing W, we could equivalently maximize a monotonic increasing function of W, like, oh, say, $\frac{1}{n}\log W$:

$$\begin{aligned} \frac{1}{n}\log W &= \frac{1}{n}\log\frac{n!}{n_1!\cdots n_m!} \\ &= \frac{1}{n}\log\frac{n!}{np_1!\cdots np_m!} \\ &= \frac{1}{n}(\log n! - \sum_i \log np_i!) \end{aligned}$$

Wallis derivation

- Now, we can bring in *Stirling's approximation*:

$$\begin{aligned} \log n! &= \sum_{k=1}^{n} \log k \\ &\approx \int_1^n \log x \, dx \\ &= n\log n - n + 1 \\ &\approx n\log n - n \end{aligned}$$

Wallis derivation

- Put them together and we get:

$$\begin{aligned} \frac{1}{n}\log W &= \frac{1}{n}(n\log n - n - \sum_i (np_i \log np_i - np_i) \\ &= \log n - \sum_i p_i \log np_i \\ &= \log n - (\sum_i p_i \log n + \sum_i p_i \log p_i) \\ &= \log n - (\sum_i p_i \log n + \sum_i p_i \log p_i) \\ &= (1 - \sum_i p_i)\log n - \sum_i p_i \log p_i \\ &= -\sum_i p_i \log p_i \end{aligned}$$

A simple example

- Suppose a fast-food restaurant sells \$1.00 burgers and \$2.00 chicken sandwiches. Customers, on average, pay \$1.75 for lunch. What's the probability that someone ordered a burger?

- We know:

$$\begin{aligned} P(b) + P(c) &= 1 \\ (\$1.00 \times P(b)) + (\$2.00 \times P(c)) &= \$1.75 \end{aligned}$$

- So, we can conclude:

$$\begin{aligned} (\$1.00 \times P(b)) + (\$2.00 \times (1 - P(b))) &= \$1.75 \\ P(b) &= \$0.25 \end{aligned}$$

A simple example

- Now suppose this fast-food restaurant also sells \$3.00 fish sandwiches. If customers pay \$1.75 for lunch on average, what's the probability that someone ordered a burger?

- We know:

$$\begin{aligned} P(b) + P(c) + P(f) &= 1 \\ (\$1.00 \times P(b)) + (\$2.00 \times P(c)) + (\$3.00 \times P(f)) &= \$1.75 \end{aligned}$$

- Now we have three unknown probabilities and only two constraints.

- Out of the many possible ways of assigning probabilities, we want to find the one that maximizes the entropy.

A simple example

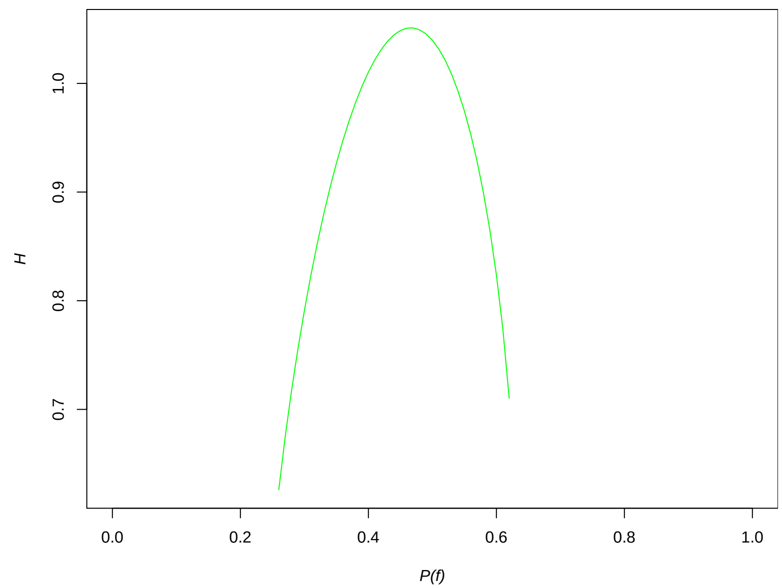
- We can use the constraints to eliminate two of the unknowns:

$$\begin{aligned} P(c) &= -2\,P(b) + 1.25 \\ P(f) &= P(b) - 0.25 \end{aligned}$$

- Now we can apply MaxEnt:

$$\begin{aligned} H = & -P(b)\log P(b) - \\ & (-2\,P(b) + 1.25)\log(-2\,P(b) + 1.25) - \\ & (P(b) - 0.25)\log(P(b) - 0.25) \end{aligned}$$

A simple example



A simple example

- To find the value of $P(b)$ which maximizes H, we take the derivative of H:

$$\frac{d}{dP(b)}H = -\log(P(b)) + 2\log(-2P(b) + 1.25) - \log(P(b) - 0.25)$$

and solve:

$$\begin{aligned} -\log(P(b)) + 2\log(-2P(b) + 1.25) - \log(P(b) - 0.25) &= 0 \\ P(b) &= 0.466 \end{aligned}$$

Maximum entropy

- Simple problems can be solved analytically, but to replace naive Bayes we need a more general solution
- We have our usual feature vector x, and we know the value of feature x_i for every instance in the training set
- From this, we can estimate the expected value $\hat{E}[x_i]$
- This gives us a set of constraints:

$$\begin{aligned} \sum P(x) &= 1 \\ \text{for each } x_i\text{: } \sum P(x)\, x_i &= \hat{E}[x_i] \end{aligned}$$

- Of the distrubutions which satisfy these constraints, we need to find the one that maximizes the entropy $H(P)$

Constrained optimization

- This is a *constrained optimization* problem: maximize a function given a set of constraints
- First, we restate the constraints:

$$\begin{aligned} 0 &= \sum P(x)\, x_i - \hat{E}[x_i] \\ 0 &= \sum P(x) - 1 \end{aligned}$$

- Next, we introduce the *Lagrangian function*:

$$\begin{aligned} \mathcal{L}(P, \lambda, \gamma) &= -\sum_x P(x) \log P(x) - \sum_i \lambda_i \left(\sum_x P(x)\, x_i - \hat{E}[x_i] \right) - \\ &\quad \gamma \left(\sum_x P(x) - 1 \right) \end{aligned}$$

Constrained optimization

- This now gives us an *unconstrained optimization* problem, which we can solve by finding the P where:

$$\nabla \mathcal{L}(P, \lambda, \gamma) = 0$$

- So, we start here:

$$\begin{aligned} 0 &= \frac{\partial}{\partial P} \mathcal{L}(P, \lambda, \gamma) \\ &= -(1 + \log P(x)) - \sum_i \lambda_i x_i - \gamma \\ \log P(x) &= -\gamma - 1 - \sum_i \lambda_i x_i \\ P(x) &= \exp(\gamma - 1) \exp \left(\sum_i \lambda_i x_i \right) \end{aligned}$$

Constrained optimization

- Recall that:

$$\begin{aligned} \sum_x P(x) &= 1 \\ &= \sum_x \exp(\gamma - 1) \exp \left(\sum_i \lambda_i x_i(x) \right) \\ \exp(\gamma - 1) &= \left(\sum_x \exp \left(\sum_i \lambda_i x_i \right) \right)^{-1} \end{aligned}$$

Constrained optimization

- Finally, substituting in $P(x)$, we get:

$$P(x) = \frac{1}{Z} \exp\left(\sum_i \lambda_i x_i\right)$$
$$Z = \sum_x \exp\left(\sum_i \lambda_i x_i\right)$$

- Parameters λ_i are chosen so that:

$$\sum_x P(x) x_i = \hat{E}[x_i]$$

- Z is sometimes called the *partition function*

MaxEnt classifiers

- The model can be constrained by anything whose expected value is interesting (e.g, presence of a word, normalized frequency of a word)
- To apply this to classification, we need the joint distribution $P(x, c)$. So, features need to be a conjunction of a *contextual predicate* and a class
- We can account for the class prior $P(c)$ by including the class itself as a feature
- Feature selection can be done in the usual way.
- Setting all features for a *baseline* class to zero will further reduce the number of features

MaxEnt classifiers

- To build a MaxEnt classifier , we need to construct a function f_i from documents to features, and then estimate λ_i for each feature i (more on that later)
- Then, to find the probability of a new document d having a class label c, we evaluate:

$$P(d, c) = \frac{\exp \sum_i \lambda_i f_i(d, c)}{\sum_{d,c} \exp \sum_i \lambda_i f_i(d, c)}$$

- Now we have a problem: the sum in the denominator ranges over *all possible documents and classes*
- One option is Monte Carlo simulation: randomly generate lots of documents according to our distribution and use them to estimate Z

MaxEnt classifiers

- Instead, we can use our training data to compute an ‘empirical’ document distribution $\tilde{P}(d)$.
- Instead of these constraints:

$$\sum_d P(d, c)\, f_i(d, c) = \sum_d \tilde{P}(d, c)\, f_i(d, c)$$

we can use these constraints:

$$\sum_d \tilde{P}(d)\, P(c|d)\, f_i(d, c) = \sum_d \tilde{P}(d, c)\, f_i(d, c)$$

- This gives us a conditional maximum entropy model:

$$P(c|d) = \frac{\exp \sum_i \lambda_i f_i(d, c)}{\sum_c \exp \sum_i \lambda_i f_i(d, c)}$$

MaxEnt classifiers

- If we are only interested in classification, then for each document we only need to find:

$$\hat{c} = \underset{c \in C}{\operatorname{argmax}} \sum_i \lambda_i f_i(d, c)$$

- This (obviously) gives us a linear decision boundary
- Since we're not summing log probabilities, there's no clear bias for longer or shorter documents
- Also known as log-linear, Gibbs, exponential, and multinomial logit models
- Other constraints yield different distributions